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4-Manifolds which embed in $\mathbb{R}^6$ but not in $\mathbb{R}^5$, and Seifert manifolds for fibered knots

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We exhibit the first-known orientable 4-manifolds which embed in $\mathbb{R}^6$ but not in $\mathbb{R}^5$. These are also the first examples of compact, spin 4-manifolds which (punctured) cannot occur as Seifert manifolds for any simple, fibered knot in S^5. Necessary and sufficient conditions are derived for a 4-manifold to embed *punctured* in $\mathbb{R}^5$, and these are applied successfully to prove positive results for several interesting classes including many of the above examples.

It has been known for some time that:

1) a closed, orientable, smooth 4-manifold X embeds in $\mathbb{R}^6$ if and only if $\omega_2(X)=\sigma(X)=0$. (Thm. 2.5 of [1], [2]);

2) X embeds punctured in $\mathbb{R}^6$ if and only if $\omega_2(X)=0$ ([2, 16]).

Curiously enough, the situation for embeddings in $\mathbb{R}^5$ was not known. In particular, it was not known if there was an orientable 4-manifold X which would embed (smoothly) in $\mathbb{R}^6$ but not in $\mathbb{R}^5$. In fact, we showed in [5] that all "simple" 4-manifolds which embed in $\mathbb{R}^6$ *do* indeed embed in $\mathbb{R}^5$. "Simple" meant geometrically simple (e.g. $S^1 \times M^3$, $F_1^2 \times F_2^2$) or algebraically simple ($H_1(X)$ the direct product of fewer than 3 cyclic groups or $\pi_1(X)$ a free product of cyclic groups). Finding obstructions to embedding in $\mathbb{R}^5$ was frustrated by the failure of the standard codimension-one obstructions (duality [10, 12] and asphericity [9]) and by the vanishing of all stable characteristic classes (since the hypotheses $\sigma(X)=\omega_2(X)=0$ imply the stable-parallelizability of X).

In this paper we exhibit homotopy obstructions to embedding in $\mathbb{R}^5$ and prove (compared to 1)) that:

Theorem 3.3. *For each odd prime p, there are closed, smooth, orientable 4-manifolds X_p with $\pi_1(X_p)\cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$ such that:*

a) *X_p embeds smoothly in $\mathbb{R}^6$,*

b) *no manifold with the homotopy-type of X_p will embed (smoothly or TOP locally-flatly) in $\mathbb{R}^5$,*

c) *$X_p \# (\#_k S^2 \times S^2)$ does not embed in $\mathbb{R}^5$.*

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As regards 2), we exhibit *necessary* and *sufficient* conditions for a closed 4-manifold to embed puctured in $\mathbb{R}^5$. However, in all cases thus far computed these conditions have *not* yielded the expected spin 4-manifold which does not embed punctured in $\mathbb{R}^5$. On the contrary:

Theorem 5.6. *If X is a closed, smooth,* spin *4-manifold with $\pi_1(X)\cong(\mathbb{Z}_p)^3$ and $p\equiv 1 \bmod 3$ then X will embed punctured in $\mathbb{R}^5$.*

This and other results show that many of the counter-examples of Theorem 3.3 do embed punctured and that a spin 4-manifold which does not embed punctured might have to be more "complicated" than these.

The manifolds of Theorem 3.3 are the first-known spin, index zero 4-manifolds which cannot be Seifert manifolds for unknotted 3-spheres in S^5 (see § 4). In fact, they cannot occur as such for any simple fibered knot, giving a partial answer to Problem 9 of [11].

The embedding of 4-manifolds in 5-space is intimately related to codimension-two knot (and link) theory in S^4 and to homology cobordism of 4-manifolds ([3–6, 8, 16]).

All manifolds and maps between them should be assumed to be smooth. However, the embedding obstructions associated to 3.3 actually obstruct TOP locally-flat embedding.

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§ 2. The obstructions

Suppose X is a closed, spin 4-manifold which embeds in $\mathbb{R}^5$ or, equivalently, S^5. For the remainder of this section suppose also that $H_1(X)\cong(\mathbb{Z}_p)^3$ for p an odd prime. What we derive below is a special case of obstructions derived in [4] (see p. 55) and [5].

If X embeds in S^5, it separates S^5 into two (stably-parallelizable) 5-manifolds A, B. Thus X is a framed-boundary (equivalent to $\sigma(X)=\omega_2(X)=0$). Furthermore, since $A\cup_X B$ is S^5, the fundamental groups of A, B are not arbitrary. It is this property, being the boundary of a 5-manifold with a certain π_1, which forms our obstruction. Let L stand for $L_p^\infty\equiv K(\mathbb{Z}_p, 1)$ and let the identification of $H_1(X)$ with $(\mathbb{Z}_p)^3$ fix a map $i\colon X\to L\times L\times L$.

Proposition 2.1. *If X (as above) embeds in S^5 then there is an epimorphism $\Psi\colon(\mathbb{Z}_p)^3\to(\mathbb{Z}_p)^2$ such that the map $\Psi_*\colon H_4(L\times L\times L)\to H_4(L\times L)$ sends $i_*([X])$ to zero.*

Proof. A Mayer-Vietoris argument implies that the inclusions induce an isomorphism:

$$\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p\xrightarrow{i_*^{-1}} H_1(X)\to H_1(A)\times H_1(B).$$

Letting A be the component with the larger H_1, it follows that there is an *epimorphism* ϕ from $\pi_1(A)$ to $\mathbb{Z}_p\times\mathbb{Z}_p$ such that the following commutes:

$$\begin{array}{ccc} & \pi_1(A) & \\ j_* \nearrow & & \searrow \phi \\ \pi_1(X) \xrightarrow{i_*} \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p & \xrightarrow{\Psi} & \mathbb{Z}_p \times \mathbb{Z}_p \end{array}$$

where $\Psi \equiv \phi \circ j_* \circ i_*^{-1}$ is an epimorphism. This induces:

$$\begin{array}{ccc} & A & \\ j \nearrow & & \searrow \phi^\# \\ X \xrightarrow{i} L \times L \times L & \xrightarrow{\Psi^\#} & L \times L \end{array}$$

from which the Proposition follows by taking H_4 of everything and noting that $j_*: H_4(X) \to H_4(A)$ is the zero map. □

Note that we could have used $H_*(\ ;\mathbb{Z}_p)$ and this would have resulted in an obstruction for any X with $H_1(X;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^3$. Furthermore, given any two such 4-manifolds X, Y and a mod p degree one map between them inducing an isomorphism on $H_1(\ ;\mathbb{Z}_p)$, this obstruction is "natural". In particular, it vanishes for X if and only if it vanishes for Y. Thus we have the philosophically interesting:

Corollary 2.2. *On the category of closed 4-manifolds with $H_1(\ ;\mathbb{Z}_p) \cong (\mathbb{Z}_p)^3$, there is a cohomology operation*

$$\theta: H^1(\ ;\mathbb{Z}_p) \to H^4(\ ;\mathbb{Z}_p \times \mathbb{Z}_p)$$

which must have a "non-trivial" zero on X if X is to embed in $\mathbb{R}^5$.

Proof. The operation θ is defined as follows. Given X and a non-zero $\lambda \in H^1(X;\mathbb{Z}_p)$, we can associate an epimorphism $\Psi: H_1(X;\mathbb{Z}_p) \to \mathbb{Z}_p \times \mathbb{Z}_p$ (this necessitates fixing certain identifications beforehand). This induces a homotopy class of maps $[f]: X \to L \times L$. If α, β are fixed generators of $H^4(L \times L;\mathbb{Z}_p)$ then let $\theta_X(\lambda)$ be the pair $(f^*(\alpha), f^*(\beta))$. □

Proposition 2.1 may be reformulated in a way that simplifies calculations by noting that any epimorphism $\Psi: (\mathbb{Z}_p)^3 \to (\mathbb{Z}_p)^2$ factors as $p \circ F$ where F is an automorphism of $(\mathbb{Z}_p)^3$ and p is the canonical projection onto the first two factors (fixing bases).

Proposition 2.3. *If X (as above) embeds in S^5, then there is an F in $GL(3,\mathbb{Z}_p)$ such that $p_* \circ F_* \circ i_*([X]) = 0$ where:*

$$H_4(X) \xrightarrow{i_*} H_4(L \times L \times L) \xrightarrow{F_*} H_4(L \times L \times L) \xrightarrow{p_*} H_4(L \times L).$$

§ 3. The manifolds

The existence of orientable 4-manifolds which embed in $\mathbb{R}^6$ but not in $\mathbb{R}^5$ will now be demonstrated in two steps. Firstly, we show that there are classes $[\alpha]$ in $H_4(L \times L \times L;\mathbb{Z})$ which map *non-zero* under each $p_* \circ F_*$ as in 2.3. Secondly, we show that these $[\alpha]$ are the images of the fundamental classes of certain spin, index zero 4-manifolds with fundamental group $(\mathbb{Z}_p)^3$.

For Step 1, certain calculations are required, these being outlined in the appendix and summarized by:

Lemma 3.1. *Given $F=(a_{ij})\in GL(3,\mathbb{Z}_p)$ and p the canonical projection to the first two factors, the map $p_*\circ F_*$: $H_4(L\times L\times L)\to H_4(L\times L)$ is given by the matrix:*

$a_{22}D_{33}$	$a_{32}D_{23}$	$a_{12}D_{33}$	$a_{12}D_{23}$	$a_{32}D_{13}$	$a_{22}D_{13}$	$a_{12}D_{13}+a_{22}D_{23}$	$a_{32}D_{33}+a_{22}D_{23}$
$a_{21}D_{33}$	$a_{31}D_{23}$	$a_{11}D_{33}$	$a_{11}D_{23}$	$a_{31}D_{13}$	$a_{21}D_{13}$	$a_{11}D_{13}+a_{21}D_{23}$	$a_{31}D_{33}+a_{21}D_{23}$

with respect to the $\mathbb{Z}_p$-bases $\{e_1,e_2,\ldots,e_6,e_7,e_9\}$ for $H_4(L\times L\times L)\cong(\mathbb{Z}_p)^8$ and $\{e_1,e_3\}$ for $H_4(L\times L)\cong(\mathbb{Z}_p)^2$ as defined in the appendix. Here D_{ij} is the determinant of the (i,j) minor of F.

Theorem 3.2. *For any odd prime p, there is a class $[\alpha]\in H_4(L\times L\times L)$ such that $p_*\circ F_*([\alpha])$ is* non-zero *for all $F\in GL(3,\mathbb{Z}_p)$.*

Proof. Let $[\alpha]$ have coordinates $(0,-r,1,0,-t,1,0,0)$ with respect to the basis $(e_1,e_2,\ldots e_6,e_7,e_9)$ where r and t depend on p and will be chosen below. Setting $p_*\circ F_*([\alpha])\equiv 0$ gives two equations over $\mathbb{Z}_p$ in the 9 variables (a_{ij}) (use 3.1). These *imply:*

$$rD_{33}D_{23}\equiv -rD_{13}^2$$
$$rD_{23}^2 \equiv D_{13}(D_{33}-tD_{23}).$$

Since F is invertible, at least one of D_{13}, D_{23}, D_{33} is non-zero. Check that if $D_{13}=D_{23}=0$ then, although the congruences above are satisfied, the original 2 equations will not be unless $D_{33}=0$. Then, assuming $r\not\equiv 0$ and making the substitution $z=D_{13}D_{23}^{-1}$, we are led to a cubic equation over $\mathbb{Z}_p$ in the one variable z:

$$z^3+tz+r\equiv 0.$$

To complete the proof we need only show that for any odd p there is a cubic of the above form which is *irreducible* in $\mathbb{Z}_p$. Since z^3-z-1 works for $p=3$, we can assume $p>3$. A simple counting argument shows that the number of irreducible, monic, cubic polynomials (z^3+az^2+bz+c) is $\frac{1}{3}p(p^2-1)$, which is greater than p^2. Since at most p^2 of these satisfy $b\equiv\frac{1}{3}a^2$, we can choose one where this does not hold. The substitution $z\to z-\frac{1}{3}a$ transforms our chosen irreducible to one of the form z^3+rz+t with r non-zero. Thus $p_*\circ F_*([\alpha])$ is non-zero for every invertible F. □

Of course there are many more such "bad" classes $[\alpha]$, as a computer search reveals. There are none, however, with only three non-zero coordinates, so the examples of Theorem 3.2 are the simplest.

For Step 2 we need to recall from the appendix that the basis $(e_1,e_2,\ldots,e_6,e_7,e_9)$ is such that $e_2=x_1\otimes x_0\otimes x_3$, $e_3=x_3\otimes x_1\otimes x_0$, $e_5=x_0\otimes x_1\otimes x_3$, $e_6=x_0\otimes x_3\otimes x_1$. The obvious map $S^1\to L$ represents x_1 while the inclusion of the 3-dimensional lens space $L(p,1)\hookrightarrow L$ represents $n_p\cdot x_3$ where n_p

is a unit. Since $x_1 \otimes x_3 \otimes x_0$ is a positive multiple of $x_1 \otimes n_p x_3 \otimes x_0$ (for example), each class $[\alpha]$ of Theorem 3.2 is the image of the fundamental class of a connected sum of copies of $S^1 \times L(p, 1)$ mapped into $L \times L \times L$ according to the recipe given by the coordinates and basis above. (Here, one should take *positive* representatives for $-r$ and $-t$ modulo p.) Let Y be this connected sum and j be this map. Add 2-handles along $Y \times \{1\} \hookrightarrow Y \times I$ to kill the kernel of j_* on H_1. With minor care, the result will be a spin cobordism from (Y, j) to (X_p, i) where $\pi_1(X_p) \cong (\mathbb{Z}_p)^3$ and $i_*([X]) = [\alpha]$. Since Y was spin and of index zero, so is X_p and, by 2.3 and 3.2, X cannot embed in $\mathbb{R}^5$.

Thus we have proved the lion's share of:

Theorem 3.3. *For each odd prime p, there are closed, smooth orientable 4-manifolds X_p with $\pi_1(X) \cong (\mathbb{Z}_p)^3$ such that:*

a) *X_p embeds smoothly in $\mathbb{R}^6$,*
b) *no manifold homotopy-equivalent to X_p embeds in $\mathbb{R}^5$,*
c) $X_p \#_k (S^2 \times S^2)$ *cannot embed in* $\mathbb{R}^5$.

Remarks. Parts b) and c) follow from 2.2 and the comments preceeding it. Thus no 4-manifold Y which possesses a degree one map (mod p) $f: Y \to X_p$ that induces an isomorphism on $H_1(\ ; \mathbb{Z}_p)$ can embed in $\mathbb{R}^5$. Proposition 2.3 actually obstructs a closed, orientable, *topological* 4-manifold from embedding TOP locally-flatly in $\mathbb{R}^5$ (uses 2.3 of [15]). We include the following "converse" to Theorem 3.3 to emphasize that, for these simple 4-manifolds, the obstruction of 2.3 is the *sole* obstruction to stable embedding in $\mathbb{R}^5$.

Theorem 3.4 (Theorem 9.6 of [4]). *If $\pi_1(X) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, and if the (orientable) X embeds in $\mathbb{R}^6$, and if $i_*([X]) = [\alpha]$* is not *one of the "bad" classes of Proposition* 2.3 (*i.e., there is $F \in GL(3, \mathbb{Z}_p)$ such that $p_* \circ F_* \circ i_*([X]) = 0$*) *then* $X \#_k (S^2 \times S^2)$ will *embed in* $\mathbb{R}^5$.

§ 4. Seifert manifolds for knots in S^5

The embedding of 4-manifolds in S^5 is related to the question of which 4-manifolds can occur as the Seifert manifolds of knotted S^3's in S^5. Problem 9 of [11] asks, "What manifolds can occur as Seifert manifolds of higher-dimensional fibered knots?" We have the following partial solutions:

Corollary 4.1. *If $\pi_1(X) = (\mathbb{Z}_p)^3$, and X fails to satisfy* 2.3 (*e.g. the X_p of* 3.3), *then X^0 is not a Seifert manifold for any fibered knot in S^5 whose closed fiber embeds in $\mathbb{R}^5$. In particular, X^0 is not a Seifert manifold for any* simple *fibered knot (including the unknot).*

Proposition 4.2. *If V^0 is an (orientable) Seifert manifold for the knot $K: S^n \hookrightarrow S^{n+2}$, $n \geq 1$, where K is a fibered knot with fiber F^0, then V embeds in $\mathbb{R}^{n+2}$ if F does.*

Proof. Since $\pi_n(\mathrm{SO}(2)) \cong 0$ for $n \geq 2$, there is a unique framing on the n-sphere K (only a preferred framing if $n=1$) such that surgery via it yields $Y^{n+2} \cong S^1 \times_\phi F$, with V embedded in Y. Thus $V \hookrightarrow F \times \mathbb{R}$ and hence in a product neighborhood of $F \hookrightarrow \mathbb{R}^{n+2}$. □

Proof. of 4.1. The first statement follows directly from 4.2 above. If X^0 were a Seifert manifold for the simple knot K with fiber F^0, then $[X\#(-F)]^0$ would be a Seifert manifold for $K\# -K$. The latter is a simple fibered knot whose fiber, $[F\# -F]^0$ is 1-connected and has index 0. Then Proposition 6.1 of [5] would imply that $F\# -F$ embeds in $\mathbb{R}^5$, and, by 4.2, so would $X\#(-F)$. But the latter possesses a degree one map to X which is an isomorphism on H_1. Thus, since X does not embed in $\mathbb{R}^5$, $X\#(-F)$ does not either (Corollary 2.2, remark after 3.3). □

Corollary 4.3. *If X is a closed, orientable, index zero 4-manifold then the following are equivalent:*

A) *X occurs as Seifert manifold of a simple fibered knot in S^5,*
B) *X occurs as Seifert manifold of the unknot in S^5,*
C) $X \hookrightarrow \mathbb{R}^5$.

Proof. A)⇒C) since $\sigma(F)=\sigma(X)=0$ so $F \hookrightarrow \mathbb{R}^5$ [5].

§ 5. Punctured embedding in $\mathbb{R}^5$

Having found 4-manifolds X which embed in $\mathbb{R}^6$ but not in $\mathbb{R}^5$ it is natural to address 2) (of the introduction) and ask, for example, if these embed after removing a small open 4-ball (called *punctured embedding* and denoted $X^0 \hookrightarrow \mathbb{R}^5$). Surprisingly, many *do*, including all X_p (i.e., those of 3.2). As of now, we do not know of a counter-example to the statement: "Every closed, smooth, spin 4-manifold embeds punctured in $\mathbb{R}^5$".

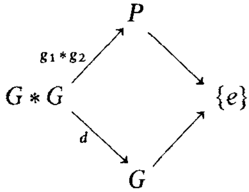
Punctured embedding is, of course, closely related to questions of which punctured 4-manifolds occur as Seifert manifolds for knotted S^3's in S^5. A counter-example to the above statement would be one which could not occur. The results of this section can be interpreted in this context.

The theorem below gives *ncessary* and *sufficient conditions* of a homotopy-theoretic nature *for X^0 to embed* in $\mathbb{R}^5$. The subsequent theorems greatly simplify these conditions for certain categories of manifolds, so that the question of punctured embedding can actually be decided. All of our calculations to date have yielded positive results (see below). However, we believe that counter-examples to the above statement should be the rule, not the exception.

If $f: G \to P$ is a group homomorphism, then f induces a homomorphism $\Gamma_k(f)$: $H_k(G;\mathbb{Z}) \to H_k(P;\mathbb{Z})$. Fixing certain identifications beforehand makes this a well-defined functor.

Theorem 5.1. *Let X be a closed, orientable 4-manifold with $\pi_1(X) \cong G$. Then X embeds punctured in $\mathbb{R}^5$ if and only if there is a finitely-presented group P and homomorphisms g_i: $G \to P$ $i=1,2$ such that:*

i) *the push-out below is trivial, where* $d(xy)=xy$ *for* $x\in G*\{e\}$ *and* $y\in\{e\}*G$



ii) $\Gamma_1(g_1)-\Gamma_1(g_2)$ *is an isomorphism* $H_1(G)\to H_1(P)$,
iii) $\Gamma_2(g_1)-\Gamma_2(g_2)$ *is an epimorphism* $H_2(G)\to H_2(P)$,
iv) *for some spin structure* σ *on* X *and maps* $g_i^{\#}: X\to K(P,1)$ *inducing the* g_i, *we have*

$$(X,\sigma,g_1^{\#})\sim(X,\sigma,g_2^{\#}) \quad \text{in } \Omega_4^{\mathrm{Spin}}(K(P,1)).$$

Furthermore, if G *is abelian, then* P *can always be chosen to be abelian (i.e.,* $P\cong G$*).*

Proof. X embedding punctured in $\mathbb{R}^5$ is equivalent to $(X\#-X)$ embedding in S^5 with one complementary component equal to $X^0\times I$. The group P will be the fundamental group of the other component. From Theorem 4.1 of [5], the above conditions are necessary to embed $(X\#-X)$ in this manner. On the other hand, by Theorem 7.3 of [5], they are sufficient to embed $(X\#-X)\#_k(S^2\times S^2)$ in S^5, from which it follows that $X^0\hookrightarrow\mathbb{R}^5$. The final assertion follows from assuming i)–iv) for *some* P, replacing it by $G\xrightarrow{g_i}P\xrightarrow{h}P/[P,P]$ and verifying that i)–iv) still hold. The only tricky part (ii) is overcome by observing that $\Gamma_1(g_1)-\Gamma_1(g_2)$ being an isomorphism implies that $\Gamma_2(h)$ is an epimorphism. □

To make effective use of 5.1, we must better understand $\Omega_4^{\mathrm{Spin}}(K(P,1))$. These are the spin bordism groups of spin 4-manifolds and maps into an Eilenberg-Maclane space (see [4, 5, 7]). We shall abbreviate these as $\Omega_4^{\mathrm{Spin}}(P)$ where P is a group. We then have the very useful:

Theorem 5.2. *If* $H_2(P;\mathbb{Z}_2)\cong H_3(P;\mathbb{Z}_2)\cong 0$, *then* $(X,\sigma,f)\in\Omega_4^{\mathrm{Spin}}(P)$ *is zero if and only if*

1) *index*$(X)=0$, *and*
2) $f_*([X])=0$ *in* $H_4(P;\mathbb{Z})$.

Proof. ($\Leftarrow$) The Atiyah-Hirzebruch spectral sequence $H_p(P;\Omega_{4-p}^{\mathrm{Spin}})\Rightarrow\Omega_4^{\mathrm{Spin}}(P)$ has vanishing E^2 terms except for $E_{0,4}^2$ and $E_{4,0}^2$ ([4, 5, 7] and [13]). In general $E_{p,q}^{\infty}\cong J_{p,q}/J_{p-1,q+1}$ where $J_{p,q}=\text{image }(\Omega_{p+q}^{\mathrm{Spin}}(K^p,K^{p-1})\to\Omega_{p+q}^{\mathrm{Spin}}(K))$. Thus $E_{0,4}^2\cong E_{0,4}^{\infty}$ is the image of the (split) monomorphism $\Omega_4^{\mathrm{Spin}}\xrightarrow{i}\Omega_4^{\mathrm{Spin}}(P)$ whose cokernel is $E_{4,0}^{\infty}\subset H_4(P;\mathbb{Z})$. Since $\sigma(X)=0$, we need only show that: $(X,\sigma)\xrightarrow{f}K\equiv K(P,1)$ can be borded into the 3-skeleton if and only if $f_*([X])=0$. We outline the "if" part. Suppose $f_*([X])=-z$, a 4-cycle in $C_4(K^4)$ (here K^4 is the 4-skeleton of $K(P,1)$), and that $z=\partial(\Delta)$ with $\Delta\in C_5(K^5)$. Then $(X,\sigma,f)\sim(X,\sigma,f)\coprod\partial(D^5,\ \text{standard},\ i\colon D^5\to\Delta)\sim(X\#S^4,\sigma,g)$ where

$g_*([X \# S^4])=0$ in $H_4(K^4; \mathbb{Z})$. Make g transverse to the barycenters of all 4-cells and pull back these points to get a collection of oriented points in X. Because of our previous adjustment, the collection will be trivial in Ω_0 and thus, by adding 1-handles to $X \times I$, we can bord (X, σ, g) to (Y, σ', h) where $h(Y)$ misses the barycenters. □

We can now prove an extraordinary consequence of Theorem 5.1, which leads immediately to positive results on punctured embeddings and could eventually produce negative results. Recall that Ω_4^{Spin} is generated by a 1-connected, spin 4-manifold of index -16 which we denote by K^4 (– the Kummer surface). Recall that p denotes an *odd* prime.

Theorem 5.3. *If X is a closed, spin 4-manifold with $\pi_1(X) \cong (\mathbb{Z}_p)^m$ and index$(X) = 16n$, then X embeds* punctured *in $\mathbb{R}^5$ if there is an $F \in GL(m, \mathbb{Z}_p)$ such that:*

a) *$\Gamma_1(F)-I$ is in $GL(m, \mathbb{Z}_p)$,*
b) *$\Gamma_2(F)-I$ is in $GL(m, \mathbb{Z}_p)$,*
c) *$\Gamma_4(F)-I$ is zero on $i_*([X]) \in H_4((\mathbb{Z}_p)^m)$.*

Furthermore, in the case $\pi_1(X) \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$, if $X \#(\#_n K^4) \#(\#_k S^2 \times S^2)$ does not embed in $\mathbb{R}^5$, then these conditions are necessary and sufficient for X to embed punctured.

Remarks. Note that $X \#(\#_n K^4)$ embeds *stably* in $\mathbb{R}^5$ (for $m=3$) if and only if $i_*([X])$ is one of the "good" classes in $H_4(L \times L \times L)$ (see 2.3 and 3.4). Since these are totally determined (and obviously imply punctured embedding), the only interesting cases *are* where the above conditions are necessary. Notice how close these conditions are to saying that X^0 is a Seifert manifold for a *fibered* knot in S^5.

Proof. Apply Theorem 5.1, choosing $P=(\mathbb{Z}_p)^m$, $g_1=F$, $g_2=I$. Use Theorem 5.2 to translate iv) into c). This proves the first part of the Theorem. The second part is more difficult. Suppose X^0 embeds in $\mathbb{R}^5$ but that $i_*([X])$ is one of the "bad" classes of 2.3, i.e., $X \#(\#_n K^4) \#(\#_k S^2 \times S^2)$ does *not* embed in $\mathbb{R}^5$. By 5.1 we have F, G in $\mathrm{Hom}(\mathbb{Z}_p^3, \mathbb{Z}_p^3)$ which satisfy

a) $F-G$ invertible
b) $\Gamma_2(F)-\Gamma_2(G)$ invertible
c) $(X, \sigma, F^\#) \sim (X, \sigma, G^\#)$ in $\Omega_4^{\mathrm{Spin}}(L \times L \times L)$.

It suffices to show that $\operatorname{rank} F = \operatorname{rank} G = 3$ (since then we can modify by an automorphism to assume that $G=I$). This is done by disqualifying all of the other cases. Assume $\operatorname{rank} F \geqq \operatorname{rank} G$. To satisfy a), we need $\operatorname{rank} F + \operatorname{rank} G \geqq 3$. Suppose $\operatorname{rank} G \leqq 1$; then $(X, \sigma, G^\#)$ would factor through $\Omega_4^{\mathrm{Spin}}(L) \cong \Omega_4^{\mathrm{Spin}}$ so $(X, \sigma, F^\#) \sim (X, \sigma, \text{constant}) \sim (\#_n(-K^4), \eta, \text{ constant})$. Thus $(X \#_n K^4, \sigma \#_n \eta, F^\#)$ would be trivial in $\Omega_4^{\mathrm{Spin}}(L \times L \times L)$. Since the rank of F would necessarily be at least 2, the obstruction of 2.1 (and 3.4) would *a fortiori* vanish for $X \#(\#_n K^4)$ implying that it embedded stably in $\mathbb{R}^5$. Therefore, we can assume rank $F \geqq$ rank $G \geqq 2$. Next, the case rank $F=$ rank $G=2$ can be eliminated because the last paragraph of our appendix reveals that:

$$\operatorname{rank} F \leqq 2 \Rightarrow \operatorname{rank} \Gamma_2(F) \leqq 1$$
$$\operatorname{rank} G \leqq 2 \Rightarrow \operatorname{rank} \Gamma_2(G) \leqq 1,$$

and b) implies rank $\Gamma_2(G)+$ rank $\Gamma_2(F) \geqq 3$. Finally, suppose that rank $G=2$ and rank $F=3$. Without loss, we can assume G is induced by the canonical "projection" to the *second* two factors, $G^{\#}: L\times L\times L \to L\times L \hookrightarrow L\times L\times L$. Then c) implies that $(X \underset{n}{\#} K^4, \sigma, F^{\#}) \sim (X \underset{n}{\#} K^4, \sigma, G^{\#})$ in $\Omega_4^{\mathrm{Spin}}(L\times L\times L)$. By composing with the canonical projection $p: L\times L\times L \to L\times L$ to the *first* two factors, it follows that $(X \underset{n}{\#} K^4, \sigma, p\circ F^{\#}) \sim (X \underset{n}{\#} K^4, \sigma, p\circ G^{\#})$. But $p\circ G^{\#}$ will factor through L so (as above) $(X \underset{n}{\#} K^4, \sigma, p\circ F^{\#}) \sim 0$ in $\Omega_4^{\mathrm{Spin}}(L\times L\times L)$, showing *a fortiori* that the obstruction of 2.3 vanishes and hence contradicting our assumption. □

Now for our particular results.

Theorem 5.4. *The 4-manifolds X_p of Theorem 3.2 embed punctured in $\mathbb{R}^5$ if $p \not\equiv 1 \bmod 3$.*

Proof. We apply Theorem 5.3. Recall $i_*([X_p])=(0, -r, 1, 0, -t, 1, 0, 0)$ in $H_4(L\times L\times L)$ where r, t are units. The hypothesis on p insures that any number is a cube modulo p (page 54 of [14]) so $r \equiv y^3$. Then,

$$F=\begin{bmatrix} 0 & 0 & y \\ y & ty^{-2} & 0 \\ 0 & y^{-2} & 0 \end{bmatrix}$$

satisfies a, b and c of 5.3. □

Theorem 5.5. *If X is any closed, spin, smooth 4-manifold with $\pi_1(X) \cong \mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$ then X embeds punctured in $\mathbb{R}^5$.*

Proof. We verified by computer that, for any $[\alpha] \in H_4(L\times L\times L)$, there is an F as required by 5.3. The calculations in the appendix are, of course, used heavily. This could be done for any p if one had the computer time. □

In the following theorem, we succeed, almost like the magicians of old, in pulling the matrix F out of thin air.

Theorem 5.6. *If X is any closed, spin, smooth 4-manifold with $\pi_1(X) \cong \mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p$ and $p \equiv 1 \bmod 3$, then X embeds punctured in $\mathbb{R}^5$.*

Proof. Apply Theorem 5.3 with $F=cI$ where c is chosen so that $c^3 \equiv 1$, $c^2 \not\equiv 1$ modulo p [14]. Notice that the functor Γ_2 is "quadratic" with respect to scalar multiplication, that is $\Gamma_2(cF)=c^2\Gamma_2(F)$. Similarly, Γ_4 is "cubic" (see Appendix). Thus $\Gamma_4(cI)-I$ is the zero matrix. □

Corollary 5.7. *All of the X_p of Theorem 3.2 embed punctured in $\mathbb{R}^5$.*

Proof. Use 5.4, 5.5 and 5.6.

We must stress that we have *not* shown that every spin X with $\pi_1(X) \cong (\mathbb{Z}_p)^3$ embeds punctured. If $p \equiv -1 \bmod 3$, we have shown this only for certain ones called X_p (see 3.2).

The gimmick of Theorem 5.6 can be only partially extended to X with $\pi_1(X) \cong (\mathbb{Z}_p)^m$ $m > 3$, because Γ_4 fails to be "cubic". Let us investigate further to see what can be salvaged.

The subspace T of $H_4((\mathbb{Z}_p)^m)$ generated by basis elements of the form $x_1 \otimes x_1 \otimes x_1 \otimes x_1 \otimes x_0 \otimes \ldots \otimes x_0$ and permutations is $GL(m, \mathbb{Z}_p)$-invariant. Therefore if $\pi_1(X) \cong (\mathbb{Z}_p)^m$ then the following makes sense:

Definition. X *is* algebraically atoroidal *if the subspace spanned by* $i_*([X]) \in H_4((\mathbb{Z}_p)^m)$ *has trivial projection on* T.

It follows that $i_*([X]) = [\alpha] \in H_4(Lx \ldots xL)$ is a sum of elements of the form $x_1 \otimes x_3 \otimes x_0 \otimes \ldots \otimes x_0$ and of the form

$$u_*^{-1}(x_2 \otimes x_1 \otimes x_1 \otimes x_0 \ldots x_0 \pm x_1 \otimes x_2 \otimes x_1 \otimes x_0 \otimes \ldots \otimes x_0).$$

These elements have the property that $\Gamma_4(cF) = c^3 \Gamma_4(F)$ on the subspace generated by them (for any $c \in \mathbb{Z}_p$ and $F \in GL(m, \mathbb{Z}_p)$). This is essentially because they are constructed using the Bockstein exactly once. Therefore, if $[\alpha]$ is atoroidal then $\Gamma_4(cI)[\alpha] = c^3[\alpha]$ so, just as in 5.7, we have:

Theorem 5.8. *If X is a closed, smooth, spin, algebraically atoroidal* 4*-manifold with $\pi_1(X) = (\mathbb{Z}_p)^m$ with $p \equiv 1 \bmod 3$, then X embeds punctured in* $\mathbb{R}^5$.

§6. Questions and problems

A) Find a closed, spin 4-manifold which does not embed punctured in $\mathbb{R}^5$?

B) If $X \# S^2 \times S^2$ embeds in $\mathbb{R}^5$ then does X necessarily embed in $\mathbb{R}^5$ (see §7 of [5])?

C) If X_p^0 embeds in S^5, can the embedding be chosen so that its image is a Seifert manifold for a *fibered* knot in S^5?

Appendix. Some calculations

Letting $L \equiv L_p^\infty \simeq K(\mathbb{Z}_p, 1)$ and choosing $S^1 \hookrightarrow L$ to represent a generator x_1 of $H_1(L; \mathbb{Z})$, we can generate $H^*(L; \mathbb{Z}_p)$ by $x^1, x^2 = \beta x^1, x^3 = x^1 \beta x^1, x^4 = \beta x^1 \beta x^1$. Here, degree is indicated by superscripts, cup product is denoted $x^1 \cup x^1 = x^1 x^1$, and β is the cohomology Bockstein associated to $1 \to \mathbb{Z}_p \to \mathbb{Z}_{p^2} \xrightarrow{\mu} \mathbb{Z}_p \to 1$. Bases of the corresponding $H_*(L; \mathbb{Z}_p)$ and $H_*(L; \mathbb{Z})$ are also induced.

Let $\gamma_1 = x^1 \otimes x^0 \otimes x^0$, $\gamma_2 = x^0 \otimes x^1 \otimes x^0$, $\gamma_3 = x^0 \otimes x^0 \otimes x^1$ be the generators of $H^1(L \times L \times L; \mathbb{Z}_p)$ (for visual clarity we will use $\otimes$ instead of cohomology or homology cross product). These generate $H^*(L \times L \times L; \mathbb{Z}_p)$ if we use β. In particular, $H^4(L \times L \times L; \mathbb{Z}_p)$ is generated as a $\mathbb{Z}_p$-vector space by 9 elements of the form $\gamma_i \gamma_j \beta \gamma_k$ where $i < j$, and 6 elements of the form $\beta \gamma_i \beta \gamma_j$ where $i \leqq j$.

Given $F \in GL(3, \mathbb{Z}_p)$, which we think of as acting on $H^1(L \times L \times L; \mathbb{Z}_p)$ by:

$$F\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix}=(a_{ij})^T\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix}=A^T\begin{bmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{bmatrix},$$

we shall compute the induced map on $H_4(L\times L\times L;\mathbb{Z})$ by first computing F^* on $H^4(L\times L\times L;\mathbb{Z}_p)$. We see that:

1) $$F^*(\gamma_i\gamma_j\beta\gamma_k)=\sum_{q=1}^{3}\sum_{m<n} a_{qk}D_{(i,j,m,n)}\gamma_m\gamma_n\beta\gamma_q$$

where $D_{(i,j,m,n)}$ is the determinant of the minor of A created by intersecting the i^{th} and j^{th} columns with the m^{th} and n^{th} rows (see pp. 239, 249, 253, 255, 281 of [17]). Similarly, the subspace generated by elements of the form $\beta\gamma_i\beta\gamma_j$ is F^*-invariant. Fix a basis of $H^4(L\times L\times L;\mathbb{Z}_p)$ by letting $\hat{e}_1=\gamma_1\gamma_2\beta\gamma_2$, $\hat{e}_2=\gamma_1\gamma_3\beta\gamma_3$, $\hat{e}_3=\gamma_1\gamma_2\beta\gamma_1$, $\hat{e}_4=\gamma_1\gamma_3\beta\gamma_1$, $\hat{e}_5=\gamma_2\gamma_3\beta\gamma_3$, $\hat{e}_6=\gamma_2\gamma_3\beta\gamma_2$, $\hat{e}_7=\gamma_2\gamma_3\beta\gamma_1$, $\hat{e}_8=\gamma_1\gamma_3\beta\gamma_2$, $\hat{e}_9=\gamma_1\gamma_2\beta\gamma_3$ and letting $\hat{e}_{10}$ through $\hat{e}_{15}$ be the other six generators. With respect to this basis,

$$F^*=\left[\begin{array}{c|c} C^T & 0\\ \hline 0 & B^T\end{array}\right]$$

where C^T is a 9×9 matrix derived from 1) and B^T is an unspecified 6×6 matrix. The matrix of F_* on $H_4(L\times L\times L;\mathbb{Z}_p)$ with respect to the dual basis $\{\bar{e}_i\}$ is the transpose of the above.

We are interested in the image of the natural map

$$\mu_*: H_4(L\times L\times L;\mathbb{Z})\to H_4(L\times L\times L;\mathbb{Z}_p).$$

This is exactly the kernel of the homology Bockstein β_4 associated to the same coefficient sequence as above (p. 281 of [17]). It may thus be determined that the image of μ_* is generated by

$$\bar{e}_1,\ \bar{e}_2,\ \bar{e}_3,\ \bar{e}_4,\ \bar{e}_5,\ \bar{e}_6,\ \bar{e}_7+\bar{e}_8,\ \bar{e}_9+\bar{e}_8$$

(pp. 255, 281 of [17]). Define a new basis

$$e_1=\bar{e}_1,\ \dots,\ e_6=\bar{e}_6,\ e_7=\bar{e}_7+\bar{e}_8,\ e_8=\bar{e}_8,\ e_9=\bar{e}_9+\bar{e}_8,\ e_{10}=\bar{e}_{10},\ \dots,\ e_{15}=\bar{e}_{15},$$

so the change of basis matrix is of the form:

$$\begin{bmatrix}R & 0\\ 0 & I\end{bmatrix}$$

where R is 9×9. Then the matrix of F_* on *the image of* μ_* is $R^{-1}CR$ *with the eighth row and column deleted.* If we choose the obvious basis (also called $\{e_i\}$) for $H_4(L\times L\times L;\mathbb{Z})$ consisting of $e_i\equiv\mu_*^{-1}(e_i)$ $i=1,2,3,4,5,6,7,9$, then *the matrix of* F_* *on* $H_4(L\times L\times L;\mathbb{Z})$ *is this same matrix* (p. 222 of [17]). Looking back, we see that

$$e_1=x_1\otimes x_3\otimes x_0,\quad e_2=x_1\otimes x_0\otimes x_3,\ \dots,\ e_6=x_0\otimes x_3\otimes x_1,$$
$$e_7=\mu_*^{-1}(x_2\otimes x_1\otimes x_1+x_1\otimes x_2\otimes x_1),\quad e_9=\mu_*^{-1}(x_1\otimes x_1\otimes x_2+x_1\otimes x_2\otimes x_1)$$

where we have used subscripts to emphasize that we are in homology.

Finally, note that the inclusion $L(p,1)\hookrightarrow L_p^\infty\equiv L$ certainly represents a generator of $H_3(L;\mathbb{Z})$ so let us call it $n_p\cdot x_3$ where n_p is a unit in $\mathbb{Z}_p$. □

We need to compute the map p_* on $H_4(\ ;\mathbb{Z})$ induced by the projection $L\times L\times L\to L\times L$ onto the first two factors. Clearly $H_4(L\times L)$ is generated over $\mathbb{Z}_p$ by $p_*(e_1)$ and $p_*(e_3)$, and in fact the map p_* is the obvious projection (p. 235 of [17]). Thus, $p_*\circ F_*$ is given by the first and third rows of $R^{-1}CR$ with the eighth column deleted. This is written out in Lemma 3.1. □

The automorphism $F_*: H_2(L \times L \times L) \hookleftarrow$ induced by F can be similarly deduced, and is given rather neatly by the transpose of the matrix whose (i, j) entry is D_{ij} (the determinant of the (i, j) minor of A). This is with respect to the basis

$$\{x_0 \otimes x_1 \otimes x_1, x_1 \otimes x_0 \otimes x_1, x_1 \otimes x_1 \otimes x_0\}. \quad \square$$

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